The Far Fields Excited by a Point Source in a Passive Dissipationless Anisotropic Uniform Waveguide*

A. D. BRESLER†

Summary—**The direction of the net power flow associated with a propagating mode of an arbitrary passive disspationless anisotropic uniform waveguide may be opposite to its direction of (phase) propagation. It is shown that when a point source is introduced into a waveguide in which this is the case, such propagating modes contribute to the fields excited by this source only in that direction for which their power flow is directed away from the source. In addition it is shown that the nonpropagating modes contribute to the total field only in that direction in which they decay with increasing distance away from the source so that the far fields are given by a superposition of propagating modes only. The proof given makes use of the known properties of the frequency dependence of the physical parameters of any linear passive system in which the causality restriction is satisfied.**

INTRODUCTION

THE motivation for the study reported here becomes evident when one attempts to formulate the solutions to discontinuity problems involving waveguides containing anisotropic media. For example, let us consider the discontinuity problem posed by the introduction of a perfectly conducting transverse obstacle into a perfectly conducting rectangular waveguide which is either empty or, as in Fig. 1, partially filled with a dissipationless anisotropic ferrite. In either case, the formulation of the solution to the problem involves replacing the obstacle by a distribution of induced currents [1] and, for all $z \neq z'$, expressing the total field in the waveguide as the sum of the specified excitation plus the fields excited by the induced sources. Thus, to solve these problems requires a knowledge of the fields excited by a point source located at $z=z'$ in the uniform waveguide with the discontinuity removed, *i.e.*, the Green's function for the uniform waveguide. Although we have argued from a specific example, it is well known that the conclusion can be generalized to the statement that the solution to any discontinuity problem requires that, in one guise or another, we introduce the Green's function for the uniform waveguide as an integrating factor. In the following we are concerned with the modal descriptions for such Green's functions. In this discussion we employ the term "mode" in a somewhat different sense from the conventional usage and we therefore first digress to clarify this point.

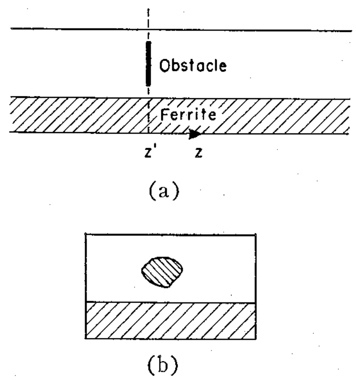

Fig. 1—Obstacle in a partially filled rectangular waveguide. (a) Longitudinal section. (b) Cross section at $z=z'$.

The modes of a uniform waveguide are taken to be those combinations of electric *and* magnetic fields which, *together*, comprise admissible solutions to the homogeneous time-independent Maxwell equations. These equations constitute an eigenvalue problem for the determination of the modes and of their associated propagation constants, or eigenvalues, κ [2]–[4]. Isotropic uniform waveguides are reflection symmetric; *i.e.*, they are invariant to a reflection through any plane transverse to z, and therefore if κ is an eigenvalue then $-\kappa$ must also be an eigenvalue. The manner in which we have defined a mode implies that we must distinguish between the modes associated with $\pm\kappa$. In isotropic waveguides the mode associated with $-\kappa$ may be obtained from that for $+\kappa$ by simply reversing the direction of the transverse magnetic field. Thus, for example, we describe an empty rectangular waveguide propagating only the TE_{10} mode as one which supports two propagating modes which propagate in opposite directions with equal magnitude propagation constants. The time and z dependence of a mode are taken as $\exp i(\kappa z - \omega t)$ so that the propagating modes are associated with real propagation constants.

In the discussion which follows we contrast certain properties of "conventional" and "anisotropic" waveguides. We define a "conventional" waveguide as a uniform waveguide completely filled with a passive dissipationless isotropic homogeneous medium and bounded (if at all) by electric walls. By an "anisotropic" waveguide we mean a uniform waveguide containing passive anisotropic media and bounded (if at all) by a combination of electric, magnetic, and other reactive walls. Uniform waveguides which are both dissipationless and "anisotropic" comprise the entire class of passive dissipationless uniform waveguides [2], [3].

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† Microwave Res. Inst., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y.

The modal description for the fields produced in a conventional waveguide by a point source at $z=z'$ distinguishes between the two regions $z \gtrless z'$ as follows. In each of these two regions the total field is described in terms of only those cut-off modes which decay in the direction away from the source and those propagating modes which propagate away from the source, not in terms of all the modes. In conventional waveguides the direction of the net power flow associated with a propagating mode coincides with its direction of (phase) propagation. Thus, for $z \gtrless z'$, the far fields at $|z-z'| \to \infty$ consist of superpositions of propagating modes which, individually, propagate and transport energy in the direction away from the source. The cut-off modes of a conventional waveguide are characterized by purely imaginary κ. Thus they are properly described as nonpropagating and one is not surprised to find that, considered individually, there is no net power flow associated with these modes.

In dissipationless anisotropic waveguides the cut-off modes need no longer be characterized by imaginary κ only, but may be associated with complex κ which must occur in complex conjugate pairs. The cut-off modes associated with complex κ are, strictly speaking, no longer describable as nonpropagating modes. Despite this it can be shown that, taken individually, there is no net power flow associated with these modes [2], [4]. Since the cut-off modes again contribute only to the energy stored in the waveguide, we expect (and will prove) that only those cut-off modes are excited which decay exponentially in the direction away from the source. Thus the far fields at $|z-z'| \to \infty$ will again consist of a superposition of propagating modes only.

For a dissipationless anisotropic waveguide which is not reflection symmetric[1] the real eigenvalues associated with the propagating modes need not occur in positive and negative pairs. In an extreme case, one might find that all the propagating modes have positive propagation constants. It does not seem likely that all these modes will contribute only for $z>z'$ and that the far field for $z<z'$ will vanish. Indeed, with this assumption, an ideal system can be constructed which violates basic thermodynamic principles [5]. The key to the riddle is supplied by the recognition that, in anisotropic waveguides, the direction of the net power flow associated with a propagating mode may be opposite to its direction of (phase) propagation. Now, while it must certainly be true that the net power flow associated with the total far fields must always be directed outward from the source, this alone does not automatically imply that each propagating mode which contributes to the far field must, individually, carry power away from the source. However, it appears intuitively obvious (at least to the author) that it must be the direction of energy transport, not propagation, which is significant in determining the direction in which a given propagating mode contributes. In the following we prove that this is indeed the case; *i.e.*, when a point source is introduced into a passive dissipationless anisotropic waveguide, the far fields in either direction away from the source consist of only those propagating modes which transport energy in that direction.

[1] Waveguides containing media with purely transverse anisotropy (*e.g.*, gyrotropic media magnetized parallel to the waveguide axis) retain the reflection symmetry property [4].

The Green's Function for an Arbitrary Uniform Waveguide

The steady-state Maxwell equations are written here in terms of linear operators $\mathfrak{L}$ and $\mathbf{\Gamma}_z$, in a form which displays explicitly the dependence on the longitudinal coordinate z only, as follows [2], [3]:

$$\left(\mathfrak{L} - \frac{1}{i}\frac{\partial}{\partial z}\Gamma_z\right)\mathbf{\Psi}(z) = -i\mathbf{\Phi}(z) \tag{1}$$

where

$$\mathfrak{L} = \omega W - \Gamma_{pt} \to \omega\begin{pmatrix}\epsilon & 0\\ 0 & \mu\end{pmatrix} - \begin{pmatrix}0 & \nabla_t \times 1\\ \nabla_t \times 1 & 0\end{pmatrix}, \tag{2}$$

$$\Gamma_z \to \begin{pmatrix}0 & i\boldsymbol{z}_0 \times 1\\ i\boldsymbol{z}_0 \times 1 & 0\end{pmatrix}, \tag{3}$$

$$\mathbf{\Psi}(z) \to \begin{pmatrix}\boldsymbol{E}(z)\\ i\boldsymbol{H}(z)\end{pmatrix}, \qquad \mathbf{\Phi}(z) \to \begin{pmatrix}\boldsymbol{J}(z)\\ i\boldsymbol{M}(z)\end{pmatrix}, \tag{4}$$

where $\boldsymbol{E}(z)$ and $\boldsymbol{H}(z)$ are the steady-state electric and magnetic fields, respectively; $\boldsymbol{J}(z)$ and $\boldsymbol{M}(z)$ represent the distributions of electric and magnetic current sources, respectively; μ and ϵ are, respectively, the (z-independent) permeability and permittivity dyadics for the media filling the guide; ∇_t is the transverse gradient operator; 1 is the unit dyadic; and $\boldsymbol{z}_0$ is the unit vector in the longitudinal direction. In (2)–(4) an arrow is used to indicate that the matrices constitute representations of abstract elements and operators in a properly defined space. The rules for operation of operators on elements or for the sequential operation of two operators consist of the normal rules of matrix algebra with the understanding that the dot product is implied for products of dyadics and vectors or of two dyadics.

The steady-state Maxwell equations (1) provide a unique specification for $\mathbf{\Psi}(z)$ only when we specify the boundary conditions on $\mathbf{\Psi}(z)$ at the waveguide walls. By the same token, $\mathfrak{L}$ is not uniquely defined unless we associate with its (formal) definition (2) a statement of the boundary conditions to be satisfied at the waveguide walls by all elements $\mathbf{\Psi}$ on which $\mathfrak{L}$ is to operate. This statement of boundary conditions constitutes a specification of a "domain" for $\mathfrak{L}$ and serves to distinguish a particular $\mathfrak{L}$ from all other operators $\mathfrak{L}$ with the same (formal) matrix representation [3].

To effect the reduction of the inhomogeneous Maxwell equations (1) to an equivalent point source excitation problem, we introduce an operator Green's function $\mathcal{G}(z, z')$ via the requirement

$$\Psi(z) = -i\int_{-\infty}^{\infty} \mathcal{G}(z, z')\Phi(z')dz'. \tag{5}$$

Assuming the validity of the required interchange of differentiation and integration operations, etc., it is evident that substitution from (5) into (1) yields:

$$\left(\mathcal{L} - \frac{1}{i}\frac{\partial}{\partial z}\Gamma_z\right)\mathcal{G}(z, z') = I\delta(z - z') \tag{6}$$

as the equation which $\mathcal{G}(z, z')$ must satisfy. In this equation $\delta(z-z')$ is the unit impulse function and I is the unit operator in the space of all coordinates still held abstract. For our purposes it is not necessary to exhibit any details of the inner structure of the (operator) Green's functions. The interested reader will find these details given in another report [3].

Thus far we have dealt with z-dependent problems. For an infinite uniform waveguide the z dependence may be eliminated by the introduction of Fourier integral representations for all z-dependent quantities. In particular, we represent $\mathcal{G}(z, z')$ as follows:

$$\mathcal{G}(z, z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{G}(\kappa)e^{i\kappa(z-z')}d\kappa \quad (\text{Im}\,\kappa = 0). \tag{7}$$

Again assuming the validity of the required interchanges of differentiation and integration operations, we substitute from (7) into (6) to obtain:

$$(\mathcal{L} - \kappa\Gamma_z)\mathcal{G}(\kappa) = I \tag{8}$$

as the equation which $\mathcal{G}(\kappa)$ must satisfy. Thus, $\mathcal{G}(\kappa)$ is seen to be the "resolvent operator" or "characteristic Green's function" for the operator $\mathcal{L}$. As such it can be shown [6] that $\mathcal{G}(\kappa)$ is analytic almost everywhere in the complex κ plane. The singularities of $\mathcal{G}(\kappa)$ are found to be poles[2] located at points in the complex κ plane corresponding to the eigenvalues of $\mathcal{L}$ [6]. Thus, in dissipationless waveguides, $\mathcal{G}(\kappa)$ may have poles on the real κ axis. When such real poles occur, the integral in (7) is not well defined. In this case, to insure the uniqueness of the transform relation, we introduce the usual "small loss approximation" by allowing ω to take on a small positive imaginary part (*i.e.*, $1 \gg \text{Im}\,\omega > 0$). When this is done, all real eigenvalues are displaced off the real axis and the integral in (7) provides a unique specification for $\mathcal{G}(z, z')$.

The Far Fields Excited by a Point Source

In the discussion which follows we confine our attention to passive dissipationless uniform waveguides. We assume that all the eigenvalues of the operator $\mathcal{L}$ associated with a waveguide of this type are known and

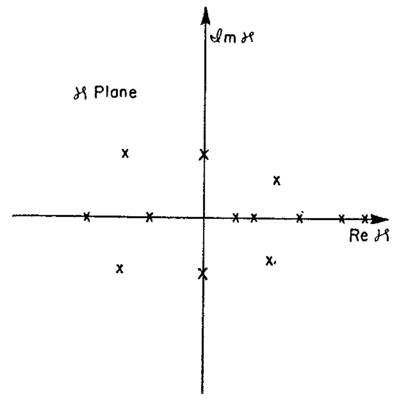

Fig. 2—Locations of poles of $\mathcal{G}(\kappa)$ when Im $\omega=0$.

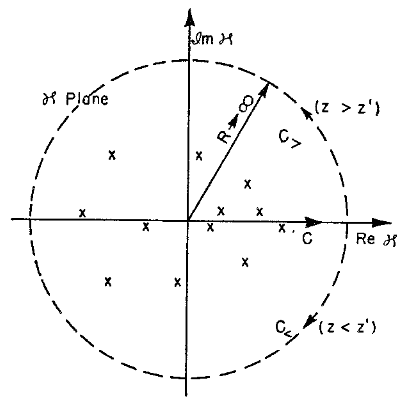

Fig. 3—Integration paths used in evaluating $\mathcal{G}(z, z')$ when $1 \gg \text{Im}\,\omega > 0$.

that these are located in the complex κ plane as indicated in Fig. 2. The disposition shown for the real propagation constants is appropriate to a waveguide which is not reflection symmetric. Only a few of the infinity of complex eigenvalues corresponding to the cutoff modes are shown in Fig. 2. These are shown to occur in complex conjugate pairs as is appropriate for a dissipationless waveguide. When we allow ω to take on a small positive imaginary part, the real eigenvalues are displaced off the real axis as indicated in Fig. 3. The complex eigenvalues are also shifted slightly. However, for sufficiently small Im ω, these remain in that half plane (above and below the real axis) in which they were located with Im $\omega=0$.

An expression for $\mathcal{G}(z, z')$ in terms of residues at the poles of $\mathcal{G}(\kappa)$ is obtained by recognizing that $\int\mathcal{G}(\kappa) \exp i\kappa(z-z')d\kappa$ taken over the path C (see Fig. 3) is identical with the closed contour integral over either $C+C_{\gtrless}$ according as $z \gtrless z'$. This is so because the behavior of $\mathcal{G}(\kappa)$ as $\kappa \rightarrow \infty$ is such that the specified integral vanishes over $C_{\gtrless}$ according as $z \gtrless z'$ [6]. Therefore, contributions to $\mathcal{G}(z, z')$ for $z \gtrless z'$ arise only from the residues at poles for which Im $\kappa \gtrless 0$. The residue series for $\mathcal{G}(z, z')$ is, for all $z \neq z'$, equivalent to a modal representation for this operator [6]. It therefore follows that, in each of the two regions $z \gtrless z'$, $\mathcal{G}(z, z')$ is completely specified by a superposition of only those modes which

[2] For waveguides with open cross sections one also finds branch point singularities. These imply the necessity of introducing branch cuts to define a unique $\mathcal{G}(\kappa)$ satisfying appropriate restrictions on the behavior of the fields in the transverse plane. Points on the branch cut correspond to eigenvalues of a continuous spectrum. For our purposes it is sufficient to view such a branch cut as a coalescence of a dense set of poles [6].

decay in the direction away from the source. This establishes that, in the limit as Im $\omega \to 0$, the far fields excited by a point source will consist of superpositions of propagating modes only.

There still remains the question of determining whether a particular propagating mode contributes to the field for $z>z'$ or $z<z'$. To answer this question we must determine whether a particular propagation constant κ_α which is real for Im $\omega=0$ has a positive or a negative imaginary part when Im $\omega>0$. This in turn requires information concerning the frequency derivative of a real propagation constant since, for $1 \gg \text{Im } \omega > 0$, we may approximate the shifted value of κ_α by:[3]

$$\kappa_\alpha(\omega \neq \omega^*) = \kappa_\alpha(\omega = \omega^*) + i\frac{d\kappa_\alpha(\omega = \omega^*)}{d\omega}\text{Im}\,\omega + \cdots. \quad (9)$$

Thus, for small positive Im ω, Im $\kappa_\alpha \gtrless 0$ according as

$$\frac{d\kappa_\alpha(\omega = \omega^*)}{d\omega} \gtrless 0.$$

This result obtains in any dissipationless waveguide. In the next section we demonstrate that, for waveguides which are both passive and dissipationless,

$$\frac{d\kappa_\alpha(\omega = \omega^*)}{d\omega}$$

is positive or negative according as the power flow associated with the corresponding propagating mode is positive or negative. This will complete the proof for the statement that in any passive dissipationless waveguide a propagating mode contributes to the field of a point source only in that direction for which the transport of energy is outward from the source.

On the Frequency Derivative of a Real Propagation Constant

To establish a connection between the power flow associated with a propagating mode and the frequency derivative of its propagation constant, we deal with the eigenvalue problem:

$$(\mathfrak{L} - \kappa_\alpha \Gamma_z)\boldsymbol{\Psi}_\alpha = 0 \quad (10)$$

which determines the z-independent mode function $\boldsymbol{\Psi}_\alpha \equiv \boldsymbol{\Psi}(\kappa_\alpha)$ associated with the eigenvalue κ_α. The eigenvalue problem (10) is obtained from (1) by setting $\boldsymbol{\Phi}(z)=0$, taking the z dependence of $\boldsymbol{\Psi}(z)$ as $\exp i\kappa_\alpha z$ and eliminating the z dependence. In this section we have occasion to deal with both z-dependent and z-independent elements. The dependence on z will be explicitly indicated for the former.

To obtain the results we are seeking, it is necessary to exploit certain properties of $\mathfrak{L}$ and Γ_z which become evident only after we have introduced an adjointness concept for operators. For this purpose we define the (hermitian) inner product [3] of two elements via:

$$(\boldsymbol{\Psi}_\beta, \boldsymbol{\Psi}_\alpha) = \iint_S [\boldsymbol{E}_\beta^* \cdot \boldsymbol{E}_\alpha + (i\boldsymbol{H}_\beta)^* \cdot (i\boldsymbol{H}_\alpha)]dS. \quad (11)$$

This definition requires an integration over the cross section, S, of the waveguide in addition to the operations usually required to obtain the (hermitian) inner product of column matrices. Based on this inner product there is associated with a given operator, *e.g.*, $\mathfrak{L}$, an adjoint operator $\mathfrak{L}^+$ via the adjointness relation [3]:

$$(\boldsymbol{\Psi}^+, \mathfrak{L}\boldsymbol{\Psi}) - (\mathfrak{L}^+\boldsymbol{\Psi}^+, \boldsymbol{\Psi}) = 0. \quad (12)$$

As it stands, this relation provides only the formal definition (matrix representation) for $\mathfrak{L}^+$. As was the case with $\mathfrak{L}$, $\mathfrak{L}^+$ is not uniquely defined without a specification of its domain. This is provided by choosing the domain of $\mathfrak{L}^+$ as the set of all elements $\boldsymbol{\Psi}^+$ so that (12) is satisfied for all elements $\boldsymbol{\Psi}$ in the domain of $\mathfrak{L}$.

$\mathfrak{L}$ is hermitian, *i.e.*, $\mathfrak{L}=\mathfrak{L}^+$, when the matrix representations and the domains of $\mathfrak{L}$ and $\mathfrak{L}^+$ are identical. It is evident from (11) and (12) that the inner product $(\boldsymbol{\Psi}, \mathfrak{L}\boldsymbol{\Psi})$ is real when $\mathfrak{L}$ is hermitian, imaginary when $\mathfrak{L}$ is skew hermitian ($\mathfrak{L}=-\mathfrak{L}^+$). It therefore follows that any operator may be expressed as the sum of hermitian and skew hermitian parts by writing, *e.g.*, $\mathfrak{L}=\mathfrak{L}_1+i\mathfrak{L}_2$ with $\mathfrak{L}_{1,2}=\mathfrak{L}_{1,2}^+$. $\mathfrak{L}^+$ is then given by $\mathfrak{L}^+=\mathfrak{L}_1-i\mathfrak{L}_2$ and it is therefore appropriate to designate $\mathfrak{L}_1=\text{Re }\mathfrak{L}$ and $\mathfrak{L}_2=\text{Im }\mathfrak{L}$. A positive hermitian operator, $\mathfrak{L}=\mathfrak{L}^+>0$, is defined as one for which $(\boldsymbol{\Psi}, \mathfrak{L}\boldsymbol{\Psi})>0$ for every $\boldsymbol{\Psi}\neq 0$ in the domain of $\mathfrak{L}$.

We turn now to a consideration of the properties of the operators $\mathfrak{L}=\omega W-\Gamma_{pt}$ and Γ_z. It is evident that $\mathfrak{L}^+=\omega^* W^+-\Gamma_{pt}^+$ and therefore, in considering $\mathfrak{L}$, we examine the properties of W and Γ_{pt} separately. Since the matrix representations in (2) and (3) for W and Γ_z, respectively, contain no differentiation operations, the domains of these operators need not be artificially restricted and it follows from (12) that the matrix representations for W^+ and Γ_z^+ are simply the conjugate transpose of those for W and Γ_z, respectively. It is further evident from ordinary matrix considerations that $\Gamma_z=\Gamma_z^+$. For an arbitrary electromagnetic field (element) $\boldsymbol{\Psi}$ we find, on substitution from (3) and (4) into (12), that:

$$(\boldsymbol{\Psi}, \Gamma_z\boldsymbol{\Psi}) = 2\,\text{Re}\iint_S z_0 \cdot \boldsymbol{E} \times \boldsymbol{H}^* dS = 2P \quad (13)$$

where P is the net power flow along $+z$ associated with the electromagnetic field. For an arbitrary z-dependent element $\boldsymbol{\Psi}(z)$ we find, on substitution from (2) and (4) into (12), that:

$$\int_{z_1}^{z_2>z_1} (\boldsymbol{\Psi}(z), W\boldsymbol{\Psi}(z))dz$$

$$= \int_{z_1}^{z_2}\iint_S [\boldsymbol{E}^*(z)\cdot\epsilon\cdot\boldsymbol{E}(z) + \boldsymbol{H}^*(z)\cdot\mu\cdot\boldsymbol{H}(z)]dSdz. \quad (14)$$

[3] To write this equation it must be assumed that κ_α is an analytic function of ω in a suitable neighborhood of the real ω axis. This assumption is justified by the results obtained in the next section.

The imaginary part of the integral on the right represents the energy/cycle dissipated in the waveguide between the two transverse planes at z_1 and z_2 and it must therefore be zero for passive dissipationless media, positive when dissipation is present. Since this must be true for any arbitrary z dependence in $\boldsymbol{\Psi}(z)$ it follows that passive dissipationless media are characterized by $W = W^+$. For passive media, dissipationless or not, we must have Re $-i\omega W \geq 0$. It is pertinent to remark that Re $-i\omega W$ is a measure of the power dissipated in the waveguide.

To evaluate $\Gamma_{pt}{}^+$ we examine

$$(\boldsymbol{\Psi}^+, \Gamma_{pt}\boldsymbol{\Psi}) - (\Gamma_{pt}\boldsymbol{\Psi}^+, \boldsymbol{\Psi}) = i \iint_S [\boldsymbol{E}^{+*} \cdot \nabla_t \times \boldsymbol{H} - \boldsymbol{H} \cdot \nabla_t \times \boldsymbol{E}^{+*} - \boldsymbol{H}^{+*} \cdot \nabla_t \times \boldsymbol{E} + \boldsymbol{E} \cdot \nabla_t \times \boldsymbol{H}^{+*}] dS \quad (15)$$

where $\boldsymbol{\Psi}$ is in the domain of Γ_{pt}. By application of the appropriate Green's identity we replace the surface integral by a line integral to obtain

$$(\boldsymbol{\Psi}^+, \Gamma_{pt}\boldsymbol{\Psi}) - (\Gamma_{pt}\boldsymbol{\Psi}^+, \boldsymbol{\Psi}) = -i \oint \boldsymbol{\nu} \cdot [\boldsymbol{E} \times \boldsymbol{H}^{+*} + \boldsymbol{E}^{+*} \times \boldsymbol{H}] ds \quad (16)$$

where $\boldsymbol{\nu}$ is the unit outward normal at the waveguide walls and s is the coordinate along the periphery of the waveguide cross section (see Fig. 4).

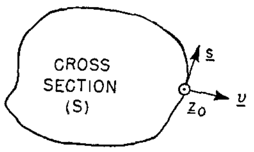

Fig. 4—Coordinates on the waveguide walls.

This last result makes evident that Γ_{pt} and $\Gamma_{pt}{}^+$ have the same matrix representations provided that the domain of $\Gamma_{pt}{}^+$ is chosen so that the boundary integral on the right vanishes identically. For complete generality we assume that $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy an impedance boundary condition at the waveguide walls.

Thus, we prescribe:

$$\boldsymbol{E} = \mathcal{Z} \cdot \boldsymbol{H} \times \boldsymbol{\nu} \text{ (on boundaries)} \quad (17)$$

where $\mathcal{Z}$ is a dyadic with components in the sz plane only. When this boundary condition is introduced into (16) it becomes evident that the domain of $\Gamma_{pt}{}^+$ is defined by the boundary condition $\boldsymbol{E}^+ = \hat{\mathcal{Z}} \cdot \boldsymbol{H} \times \boldsymbol{\nu}$ with $\hat{\mathcal{Z}} = -\tilde{\mathcal{Z}}^*$; *i.e.*, $\hat{\mathcal{Z}}$ is the negative of the conjugate transpose of $\mathcal{Z}$. Therefore, $\Gamma_{pt} = \Gamma_{pt}{}^+$ when $\mathcal{Z} = -\tilde{\mathcal{Z}}^*$, *i.e.*, when the waveguide is bounded by a combination of reactive walls. The limiting cases of electric walls ($\mathcal{Z} = 0$) and magnetic walls ($\mathcal{Z}^{-1} = 0$) are obviously included within the class of reactive walls.

For our purposes it is significant to point out that:

$$\int_{z_1}^{z_2 > z_1} \oint \boldsymbol{\nu} \cdot \boldsymbol{E}(z) \times \boldsymbol{H}^*(z) ds dz = \text{Re} \int_{z_1}^{z_2} \oint \boldsymbol{H}^*(z) \times \boldsymbol{\nu} \cdot \mathcal{Z} \cdot \boldsymbol{H}(z) \times \boldsymbol{\nu} ds dz \quad (18)$$

represents the power flow into the section of the waveguide walls lying between z_1 and z_2. The boundary integral on the right may obviously be expressed in terms of an inner product definition similar to that in (11). In this sense, we shall understand Re $\mathcal{Z}$ and Im $\mathcal{Z}$ to imply the hermitian and skew hermitian parts of $\mathcal{Z}$, respectively. Then, arguing as we did in connection with (14), we recognize that Re $\mathcal{Z} \geq 0$ for passive boundaries with Re $\mathcal{Z} = 0$ for reactive (*i.e.*, passive dissipationless) boundaries. Since Re $\mathcal{Z} = 0$ is equivalent to $\mathcal{Z} = -\tilde{\mathcal{Z}}^* = -\mathcal{Z}^+$, it follows that Γ_{pt} is hermitian for waveguides bounded by passive dissipationless walls. When we add to this the fact that $W = W^+$ for passive dissipationless media, it becomes evident that, for real ω, passive dissipationless waveguides are characterized by $\mathcal{L} = \mathcal{L}^+$.

We are now in possession of all the tools we require to establish a connection between the power flow (P_α) associated with a propagating mode ($\boldsymbol{\Psi}_\alpha$) of a passive dissipationless waveguide and the frequency derivative of its real propagation constant (κ_α). We first show that, in general, the product $\kappa_\alpha P_\alpha$ need not be positive. To do this we form the inner product of (10) with $\boldsymbol{\Psi}_\alpha$ to obtain:[4]

$$\kappa_\alpha = \frac{(\boldsymbol{\Psi}_\alpha, \mathcal{L}\boldsymbol{\Psi}_\alpha)}{(\boldsymbol{\Psi}_\alpha, \Gamma_z \boldsymbol{\Psi}_\alpha)} = \frac{(\boldsymbol{\Psi}_\alpha, \mathcal{L}\boldsymbol{\Psi}_\alpha)}{2P_\alpha}. \quad (19)$$

Thus κ_α and P_α must have the same sign, *i.e.*, the directions of propagation and of energy transport must coincide, only when $\mathcal{L} > 0$. In general, $\mathcal{L}$ is not positive and therefore $\kappa_\alpha P_\alpha$ is not necessarily positive. To obtain an expression for the derivative of κ_α we first differentiate (10) to obtain:

$$(\mathcal{L}' - \kappa_\alpha' \Gamma_z)\boldsymbol{\Psi}_\alpha + (\mathcal{L} - \kappa_\alpha \Gamma_z)\boldsymbol{\Psi}_\alpha' = 0 \quad (20)$$

where the prime superscript indicates differentiation with respect to real ω. Forming the inner product of this equation with $\boldsymbol{\Psi}_\alpha$ we obtain

$$\kappa_\alpha' = \frac{(\boldsymbol{\Psi}_\alpha, \mathcal{L}'\boldsymbol{\Psi}_\alpha) + (\boldsymbol{\Psi}_\alpha, \mathcal{L}\boldsymbol{\Psi}_\alpha') - \kappa_\alpha(\boldsymbol{\Psi}_\alpha, \Gamma_z \boldsymbol{\Psi}_\alpha')}{2P_\alpha}. \quad (21)$$

Since the domain of $\mathcal{L}$ is defined by the boundary condition (17), the boundary condition satisfied by $\boldsymbol{\Psi}_\alpha'$ is:

$$\boldsymbol{E}_\alpha' = \mathcal{Z} \cdot \boldsymbol{H}_\alpha' \times \boldsymbol{\nu} + \mathcal{Z}' \cdot \boldsymbol{H}_\alpha \times \boldsymbol{\nu} \quad (22)$$

so that $\boldsymbol{\Psi}_\alpha'$ does not lie in the domain of $\mathcal{L}$ unless $\mathcal{Z}' = 0$.

[4] Since $\mathfrak{L}$ and Γ_z are both hermitian, κ_α is given here as the ratio of two real numbers. This expression must, however, be valid for all κ_α, real or complex. This apparent contradiction is resolved by recognizing that, for Im $\kappa_\alpha \neq 0$, both numerator and denominator are equal to zero [2], [4].

Since $\mathfrak{L}=\mathfrak{L}^+$, $\Gamma_z=\Gamma_z^+$, and $\kappa_\alpha=\kappa_\alpha^*$ it follows that, for $\mathfrak{Z}'=0$, *i.e.*, for frequency independent boundary conditions:

$$(\boldsymbol{\Psi}_\alpha, \mathfrak{L}\boldsymbol{\Psi}_\alpha') = (\mathfrak{L}\boldsymbol{\Psi}_\alpha,\boldsymbol{\Psi}_\alpha') = \kappa_\alpha(\boldsymbol{\Psi}_\alpha, \Gamma_z\boldsymbol{\Psi}_\alpha') \quad (23)$$

so that the last two terms in the numerator of (21) cancel each other. When $\mathfrak{Z}'\neq 0$, the properties $W=W^+$ and $\mathfrak{Z}=-\mathfrak{Z}^+$ still obtain. These may then be employed, together with (16), (17), and (22), to establish that:

$$(\boldsymbol{\Psi}_\alpha, \mathfrak{L}\boldsymbol{\Psi}_\alpha') - (\mathfrak{L}\boldsymbol{\Psi}_\alpha,\boldsymbol{\Psi}_\alpha') = \oint \boldsymbol{H}_\alpha^* \times \boldsymbol{\nu}\cdot i\mathfrak{Z}'\cdot \boldsymbol{H}_\alpha \times \boldsymbol{\nu} ds. \quad (24)$$

Therefore, in general, the expression for κ_α' in (21) becomes:

$$2P_\alpha\kappa_\alpha' = (\boldsymbol{\Psi}_\alpha, \mathfrak{L}'\boldsymbol{\Psi}_\alpha) + \oint \boldsymbol{H}_\alpha^* \times \boldsymbol{\nu}\cdot i\mathfrak{Z}'\cdot \boldsymbol{H}_\alpha \times \boldsymbol{\nu} ds. \quad (25)$$

To establish that $P_\alpha\kappa_\alpha'>0$ we must now show that $\mathfrak{L}'>0$ and $i\mathfrak{Z}'>0$. Since Γ_{pt} is independent of ω, the requirement $\mathfrak{L}'>0$ is equivalent to $(\omega W)'>0$.

To complete our proof we will invoke results established by Wu [7], Toll [8], and Youla [9]. These authors have studied the properties of "impedance operators" $Z(\omega)$ associated with linear passive systems which satisfy the causality restriction and in which a real excitation gives rise to a real response. In essence, to qualify as an "impedance operator," $Z(\omega)$ must be a linear operator which determines the response of such a system and, for real ω, the real (hermitian) part of $Z(\omega)$ must be a measure of the dissipation of the system so that Re $Z(\omega)\geq 0$ for all $\omega=\omega^*$. The authors cited above have shown that: 1) $Z(\omega)$ is analytic and Re $Z(\omega)\geq 0$ in the upper half plane (Im $\omega>0$); 2) the limit of $Z(\omega)$ as Im $\omega\rightarrow 0$ exists from above almost everywhere on the real ω axis; and 3) $Z(\omega)=Z^+(\omega)$ for $\omega=-\omega^*$. In addition, Youla [9] has shown that when $Z(\omega)=-Z^+(\omega)$ for $\omega=\omega^*$, the derivative of Im $Z(\omega)$ on the real ω axis is always negative.[5] Now, it follows from the identifications based on (14) and (18) that both $-i\omega W$ and $\mathfrak{Z}$ qualify as impedance operators and, in the dissipationless case, both are skew hermitian for real ω. Therefore $(\omega W)'>0$ and $i\mathfrak{Z}'>0$ so that:

$$\kappa_\alpha' P_\alpha > 0. \quad (26)$$

[5] This statement ignores the possibility that $Z(\omega)$ may represent the impedance of, *e.g.*, a "short circuit" so that the derivative of Im $Z(\omega)$ on the real ω axis is equal to zero for all real ω.

This completes the proof for the statement that in any passive dissipationless waveguide a propagating mode contributes to the field excited by a point source only in that direction for which the power flow is directed outward from the source.

It is of interest to note that it follows from (26) that κ_α' cannot equal zero unless P_α is infinite. We therefore do not expect to find maximum or minimum points of $\kappa_\alpha(\omega)$ at any finite value of κ_α. This interpretation also applies to modes of the continuous spectrum of a waveguide with open cross section. In this case both numerator and denominator of (21) are proportional to delta functions so that κ_α' cannot be zero unless P_α has a higher order infinity, *i.e.*, unless the coefficient of the delta function in P_α is itself infinite. It is reasonable to expect that this will not occur at a finite κ_α.

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